

A SIMPLE UNIFORM APPROACH TO COMPLEXES ARISING FROM FORESTS

MARIO MARIETTI AND DAMIANO TESTA

ABSTRACT. In this paper we present a unifying approach to study the homotopy type of several complexes arising from forests. We show that this method applies uniformly to many complexes that have been extensively studied.

1. INTRODUCTION

In the recent years several complexes arising from forests have been studied by different authors with different techniques (see [BM], [BLN], [EH], [E], [Kl], [K1], [K2], [MT], [W]). The interest in these problems is motivated by applications in different contexts, such as graph theory and statistical mechanics ([BK], [BLN], [J]). We introduce a unifying approach to study the homotopy type of several of these complexes. With our technique we obtain simple proofs of results that were already known as well as new results. These complexes are wedges of spheres of (possibly) different dimensions and include, for instance, the complexes of directed trees, the independence complexes, the dominance complexes, the matching complexes, the interval order complexes. In all cases our method provides a recursive procedure to compute the exact homotopy type of the simplicial complex. The dimensions of the spheres arising with these constructions are often strictly related to classical graph theoretical invariants of the underlying forest. Thus we give a topological interpretation to these well-known combinatorial invariants.

Section 2 is devoted to notation and background. In Section 3 we introduce the two basic concepts of this paper: the simplicial complex property of being a grape and the strictly related notion of domination between vertices of a simplicial complex. In Section 4 we discuss several applications of these notions.

2. NOTATION

Let $G = (V, E)$ be a graph (finite undirected graph with no loops or multiple edges). For all $S \subset V$, let $N[S] := \{w \in V \mid \exists s \in S, \{s, w\} \in E\} \cup S$ be the *closed neighborhood* of S ; when $S = \{v\}$, then we let $N[v] = N[\{v\}]$. If $S \subset V$, then $G \setminus S$ is the graph obtained by removing from G the vertices in S and all the edges having a vertex in S as an endpoint. Similarly, if $S \subset E$, then $G \setminus S$ is the graph obtained by removing from G the edges in S . A vertex $v \in V$ is a leaf if it belongs to exactly one edge. A set $D \subset V$ is called *dominating* if $N[D] = V$. A set $D \subset V$ is called *independent* if no two vertices in S are adjacent, i.e. $\{v, v'\} \notin E$ for all $v, v' \in D$. A *vertex cover* of G is a subset $C \subset V$ such that every edge of G contains a vertex of C . An *edge cover* of G is a subset $S \subset E$ such that the union of all the endpoints of the edges in S is V . A *matching* of G is a subset $M \subset E$ of pairwise disjoint edges.

We consider the following classical invariants of a graph G which have been extensively studied by graph theorists (see, for instance, [AL], [ALH], [BC], [ET], [HHS], [HY]); we let

- $\gamma(G) := \min\{|D|, D \text{ is a dominating set of } G\}$ be the *domination number* of G ;
- $i(G) := \min\{|D|, D \text{ is an independent dominating set of } G\}$ be the *independent domination number* of G ;
- $\alpha_0(G) := \min\{|C|, C \text{ is a vertex cover of } G\}$ be the *vertex covering number* of G ;
- $\beta_1(G) := \max\{|M|, M \text{ is a matching of } G\}$ be the *matching number* of G .

Recall the following well-known result of König (cf [D], Theorem 2.1.1).

Theorem 2.1 (König). *Let G be a bipartite graph. Then $\alpha_0(G) = \beta_1(G)$.*

We refer the reader to [B] or [D] for all undefined notation on graph theory.

Let X be a finite set of cardinality n .

Definition 2.2. A simplicial complex Δ on X is a set of subsets of X , called *faces*, such that, if $\sigma \in \Delta$ and $\sigma' \subset \sigma$, then $\sigma' \in \Delta$. The faces of cardinality one are called *vertices*.

We do not require that $x \in \Delta$ for all $x \in X$.

Every simplicial complex Δ on X different from $\{\emptyset\}$ has a standard geometric realization. Let W be the real vector space having X as basis. The realization of Δ is the union of the convex hulls of the sets σ , for each face $\sigma \in \Delta$. Whenever we mention a topological property of Δ , we implicitly refer to the geometric realization of Δ .

As examples, we mention the $(n-1)$ -dimensional simplex ($n \geq 1$) corresponding to the set of all subsets of X , its boundary (homeomorphic to the $(n-2)$ -dimensional sphere) corresponding to all the subsets different from X , and the boundary of the n -dimensional cross-polytope, that is the dual of the n -dimensional cube. Note that the cube, its boundary and the cross-polytope are not simplicial complexes. We note that the simplicial complexes $\{\emptyset\}$ and \emptyset are different: we call $\{\emptyset\}$ the (-1) -dimensional sphere, and \emptyset the (-1) -dimensional simplex, or the empty simplex. The empty simplex \emptyset is contractible.

Let $\sigma \subset X$ and define simplicial complexes

$$(\Delta : \sigma) := \{m \in \Delta \mid \sigma \cap m = \emptyset, m \cup \sigma \in \Delta\}$$

$$(\Delta, \sigma) := \{m \in \Delta \mid \sigma \not\subset m\}.$$

The simplicial complexes $(\Delta : \sigma)$ and (Δ, σ) are usually called link and face-deletion of σ . If $\Delta_1, \dots, \Delta_k$ are simplicial complexes on X , we define

$$\text{join}(\Delta_1, \dots, \Delta_k) := \{\cup_{m_i \in \Delta_i} m_i\}.$$

If $x, y \in X$, let

$$A_x(\Delta) := \text{join}(\Delta, \{1, x\})$$

$$\Sigma_{x,y}(\Delta) := \text{join}(\Delta, \{1, x, y\});$$

$A_x(\Delta)$ and $\Sigma_{x,y}(\Delta)$ are both simplicial complexes. If $x \neq y$ and no face of Δ contains either of them, then $A_x(\Delta)$ and $\Sigma_{x,y}(\Delta)$ are called respectively the *cone on Δ with apex x* and the *suspension of Δ* . If $x \neq y$ and $x' \neq y'$ are in X and are not contained in any face of Δ , then the suspensions $\Sigma_{x,y}(\Delta)$ and $\Sigma_{x',y'}(\Delta)$ are

isomorphic; hence in this case sometimes we drop the subscript from the notation. It is well-known that if Δ is contractible, then $\Sigma(\Delta)$ is contractible, and that if Δ is homotopic to a sphere of dimension k , then $\Sigma(\Delta)$ is homotopic to a sphere of dimension $k + 1$. Note that for all $x \in X$ we have

$$(2.1) \quad \Delta = A_x(\Delta : x) \cup_{(\Delta : x)} (\Delta, x).$$

We recall the notions of collapse and simple-homotopy (see [C]). Let $\sigma \supset \tau$ be faces of a simplicial complex Δ and suppose that σ is maximal and $|\tau| = |\sigma| - 1$ (i.e. τ has codimension one in σ). If σ is the only face of Δ properly containing τ , then the removal of σ and τ is called an *elementary collapse*. If a simplicial complex Δ' is obtained from Δ by an elementary collapse, we write $\Delta \succ \Delta'$. When Δ' is a subcomplex of Δ , we say that Δ *collapses onto* Δ' if there is a sequence of elementary collapses leading from Δ to Δ' .

Definition 2.3. Two simplicial complexes Δ and Δ' are *simple-homotopic* if they are equivalent under the equivalence relation generated by \succ .

It is clear that if Δ and Δ' are simple-homotopic, then they are also homotopic, and that a cone collapses onto a point.

3. DOMINATION AND GRAPES

In this section we introduce the notions of grape and domination between vertices of a simplicial complex Δ , and we give some consequences on the topology of Δ .

Definition 3.1. A simplicial complex Δ is a *grape* if

- (1) there is $a \in X$ such that $(\Delta : a)$ is contractible in (Δ, a) and both (Δ, a) and $(\Delta : a)$ are grapes, or
- (2) Δ has at most one vertex.

Note that if Δ is a cone with apex b , then Δ is a grape; indeed for any vertex $a \neq b$ we have that both (Δ, a) and $(\Delta : a)$ are cones with apex b , thus $(\Delta : a)$ is contractible in (Δ, a) and we conclude by induction.

Proposition 3.2. *If Δ is a grape, then Δ is contractible or homotopic to a wedge of spheres.*

Proof. Proceed by induction on the number n of vertices of Δ . If $n \leq 1$, then the result is clear. If $n \geq 2$, by definition of a grape, there is a vertex a such that $(\Delta : a)$ is contractible in (Δ, a) . By equation (2.1) and [Ha, Proposition 0.18] we deduce that $\Delta \simeq (\Delta, a) \vee \Sigma(\Delta : a)$. Thus the result follows by induction on the number of vertices of Δ from the definition of grape. \square

In fact we proved that if $a \in X$ and $(\Delta : a)$ is contractible in (Δ, a) , then $\Delta \simeq (\Delta, a) \vee \Sigma(\Delta : a)$. As a consequence, if Δ is a grape, keeping track of the elements a of Definition 3.1, we have a recursive procedure to compute the number of spheres of each dimension of the wedge.

In order to prove that a simplicial complex Δ is a grape we need to find a vertex a such that $(\Delta : a)$ is contractible in (Δ, a) ; in the applications we will prove the stronger statement that there is a cone C such that $(\Delta : a) \subset C \subset (\Delta, a)$ (or equivalently if $A_b(\Delta : a) \subset (\Delta, a)$). In the two extreme cases $C = (\Delta, a)$ or $C = (\Delta : a)$, we have $\Delta \simeq \Sigma(\Delta : a)$ or $\Delta \simeq (\Delta, a)$ respectively (in the latter case Δ collapses onto (Δ, a)).

Definition 3.3. Let $a, b \in X$; a *dominates* b in Δ if there is a cone C with apex b such that $(\Delta : a) \subset C \subset (\Delta, a)$.

In the special case in which $C = (\Delta, a)$ we obtain [MT, Definition 3.4].

4. APPLICATIONS

In this section we use the concepts introduced in Section 3 to study simplicial complexes associated to forests. We shall see that all these complexes are grapes (and hence they are homotopic to wedges of spheres) by giving in each case the graph theoretical property corresponding to domination.

4.1. Oriented forests. We study the simplicial complex of oriented forests of a multidigraph. In the case of directed graphs, this concept coincides with the one introduced in [K1] by D. Kozlov (following a suggestion of R. Stanley) who called it the complex of directed trees. The reason to generalize this notion to multidigraphs is to allow an inductive procedure to work.

A multidigraph G is a pair (V, E) , where V is a finite set of elements called vertices and $E \subset V \times V \times \mathbb{N}$ is a finite set of elements called edges. If $(x, y, n) \in E$ we write $x \rightarrow_n y$, or simply $x \rightarrow y$ when no confusion is possible, and call it an edge with source x and target y , or more simply an edge from x to y . We usually identify $G = (V, E)$ with $G' = (V', E')$ if there are two bijections $\varphi : V \rightarrow V'$ and $\psi : E \rightarrow E'$ such that $\psi(x, y, n) = (\varphi(x), \varphi(y), n')$, for some $n' \in \mathbb{N}$. A multidigraph $H = (V', E')$ is a subgraph of G if $V' \subset V$ and $E' \subset E$. A directed graph is a multidigraph such that distinct edges cannot have both same source and same target. We associate to a multidigraph $G = (V, E)$ its underlying undirected graph G^u with vertex set V and where x, y are joined by an edge in G^u if and only if $x \rightarrow y$ or $y \rightarrow x$ are in E .

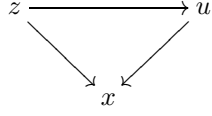
An oriented cycle of G is a connected subgraph C of G such that each vertex of C is the source of exactly one edge and target of exactly one edge. An oriented forest is a multidigraph F such that F contains no oriented cycles and different edges have distinct targets.

Definition 4.1. The complex of oriented forests of a multidigraph $G = (V, E)$ is the simplicial complex $OF(G)$ whose faces are the subsets of E forming oriented forests.

If e is a loop, i.e. an edge of G with source equal to its target, then $OF(G) = OF(G \setminus \{e\})$. Thus, from now on, we ignore the loops.

It follows from the definitions that the complex $OF(G)$ is a cone with apex $y \rightarrow x$ if and only if $y \rightarrow x$ is the unique edge with target x and there are no oriented cycles in G containing $y \rightarrow x$. For any $z \rightarrow_n u \in E$, the simplicial complex $(OF(G), z \rightarrow_n u)$ is the complex of oriented forests of the multidigraph $(V, E \setminus \{z \rightarrow_n u\})$.

We denote by $G_{\downarrow z \rightarrow u}$ the multidigraph obtained from G by first removing the edges with target u , and then identifying the vertex z with the vertex u . The reason for introducing this multidigraph is that $(OF(G) : z \rightarrow u) = OF(G_{\downarrow z \rightarrow u})$. Indeed no face of $(OF(G) : z \rightarrow u)$ contains an arrow with target u or becomes an oriented cycle by adding $z \rightarrow u$; thus there is a correspondence between the faces of the two complexes. We note that if G is a directed graph, then $G_{\downarrow z \rightarrow u}$ could be a multidigraph which is not a directed graph.

A directed graph G The multidigraph $G_{\downarrow z \rightarrow u}$

Lemma 4.2. *Let $z \rightarrow u$ and $y \rightarrow x$ be distinct vertices of $OF(G)$; then $z \rightarrow u$ dominates $y \rightarrow x$ in $OF(G)$ if and only if one of the following is satisfied:*

- $z = y$ and $u = x$;
- $u = x$ and there are no oriented cycles containing $y \rightarrow x$;
- $u \neq x$, $y \rightarrow x$ is the unique edge with target x , and all oriented cycles containing $y \rightarrow x$ contain also u .

Proof. It is clear that $z \rightarrow_n u$ dominates $z \rightarrow_m u$ whenever $m \neq n$. Thus we assume that $(z, u) \neq (y, x)$.

Let $z \rightarrow u$ dominate $y \rightarrow x$ in $OF(G)$. Suppose that $u = x$. By contradiction, let C be an oriented cycle of G containing $y \rightarrow x$. Then $z \rightarrow u \notin C$ and hence the edges of $C \setminus \{y \rightarrow x\}$ are a face of $(OF(G) : z \rightarrow u)$, but the edges of C are not a face of $(OF(G), z \rightarrow u)$ and hence $(OF(G), z \rightarrow u)$ does not contain a cone with apex $y \rightarrow x$. Suppose now that $u \neq x$. Clearly there can be no edges with target x different from $y \rightarrow x$, since each of these edges forms a face of $(OF(G) : z \rightarrow u)$. Let C be an oriented cycle of G containing $y \rightarrow x$. Then the edges of $C \setminus \{y \rightarrow x\}$ are a face of $(OF(G) : z \rightarrow u)$ if and only if C does not contain the vertex u . Since the edges of C are not a face of $(OF(G), z \rightarrow u)$ we must have that u is a vertex of C .

Conversely, let σ be a face of $(OF(G) : z \rightarrow u)$. We need to show that $\sigma \cup \{y \rightarrow x\}$ is a face of $(OF(G), z \rightarrow u)$: equivalently we need to show that it is a face of $OF(G)$, since σ does not contain $z \rightarrow u$. We may assume that $y \rightarrow x \notin \sigma$. Suppose first that $u = x$ and there are no oriented cycles containing $y \rightarrow x$; σ contains no edge with target x , since $\sigma \in (OF(G) : z \rightarrow u)$ and $\sigma \cup \{y \rightarrow x\}$ is a face of $OF(G)$ since there are no oriented cycles containing $y \rightarrow x$. Suppose now that $u \neq x$, $y \rightarrow x$ is the unique edge with target x , and all oriented cycles containing $y \rightarrow x$ contain also u . By assumption no edge of σ has x as a target; moreover if C is a cycle containing $y \rightarrow x$, then σ cannot contain all the edges of $C \setminus \{y \rightarrow x\}$, since one of these edges has target u and so it is not a face of $(OF(G) : z \rightarrow u)$. \square

We call a multidigraph F a *multidiforest* if its underlying graph F^u is a forest. The following result determines the homotopy types of the complexes of oriented forests of multidiforests.

Theorem 4.3. *Let F be a multidiforest. Then $OF(F)$ is a grape.*

Proof. Proceed by induction on the number of edges of F . It suffices to show that F contains two distinct edges $z \rightarrow u$ and $y \rightarrow x$ such that $z \rightarrow u$ dominates $y \rightarrow x$, since both $F \setminus \{z \rightarrow u\}$ and $F_{\downarrow z \rightarrow u}$ are multidiforests.

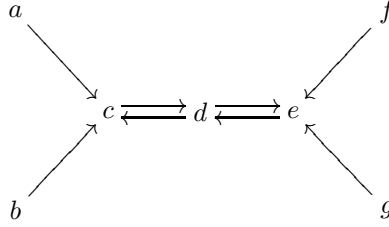
If $x \rightarrow_n y, x \rightarrow_m y$ are distinct edges, then $x \rightarrow_n y$ dominates $x \rightarrow_m y$ (and conversely) by Lemma 4.2. Thus we may assume that F is a directed graph. Let y be a leaf of F^u and let x be the vertex adjacent to y . Recall that the complex $OF(G)$ is a cone with apex $a \rightarrow b$ if and only if $a \rightarrow b$ is the unique edge with target b and there are no oriented cycles in F containing $a \rightarrow b$ (i.e. there is no edge with source b and target a). Since a cone is a grape, we only need to consider two cases:

- (1) $y \rightarrow x$ and $x \rightarrow y$ are both edges of F ,
- (2) $y \rightarrow x$ is an edge of F , $x \rightarrow y$ is not and there is $z \rightarrow x$ with $z \neq y$.

By Lemma 4.2, in case (1) $y \rightarrow x$ dominates $x \rightarrow y$, in case (2) $z \rightarrow x$ dominates $y \rightarrow x$; in both cases we conclude. \square

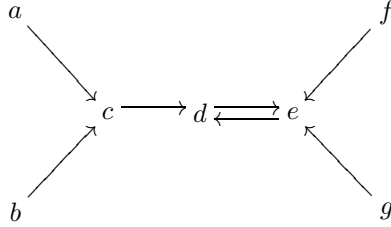
The proof of Theorem 4.3 gives a recursive procedure to compute explicitly the homotopy type of $OF(F)$, i.e. the number of spheres of each dimension. Thus it generalizes [K1, Section 4], where a recursive procedure to compute the homology groups of the complexes of oriented forests of directed trees is given.

Example 4.4. Let F be the directed tree depicted in the following figure.

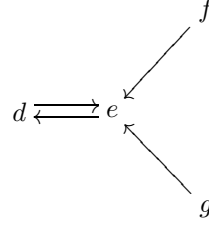


The directed tree F

By Lemma 4.2, $d \rightarrow c$ dominates $a \rightarrow c$ and hence $OF(F) \simeq OF(F_1) \vee \Sigma OF(F_2)$, where the directed trees F_1, F_2 are given in the following figure.

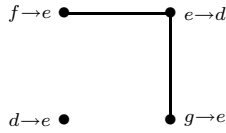


The directed tree F_1



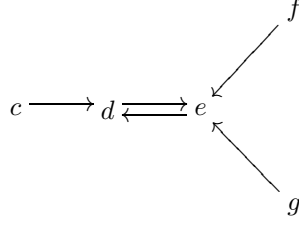
The directed tree F_2

We consider first $OF(F_2)$. The edge $d \rightarrow e$ dominates $f \rightarrow e$ in $OF(F_2)$; the complex $(OF(F_2), d \rightarrow e)$ is a cone with apex $e \rightarrow d$, and $(OF(F_2) : d \rightarrow e) = \{\emptyset\}$, since $F_2 \downarrow_{d \rightarrow e}$ has no edges different from loops. Hence $OF(F_2) \simeq S^0$ (and it is as depicted below) and $OF(F) \simeq OF(F_1) \vee S^1$.



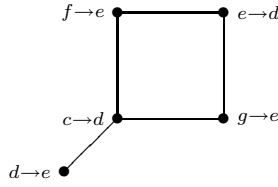
The simplicial complex $OF(F_2)$

Let us now consider $OF(F_1)$. By Lemma 4.2, $a \rightarrow c$ dominates $b \rightarrow c$. Since $(OF(F_1), a \rightarrow c)$ is a cone with apex $b \rightarrow c$, it follows that $OF(F_1) \simeq \Sigma OF(F_3)$, where F_3 is depicted in the following figure.



The directed tree F_3

The edge $e \rightarrow d$ dominates $c \rightarrow d$ in $OF(F_3)$; $(OF(F_3), e \rightarrow d)$ is a cone with apex $c \rightarrow d$, and $(OF(F_3) : e \rightarrow d)$ consists of the two isolated points $f \rightarrow e$ and $g \rightarrow e$. Thus $OF(F_3) \simeq S^1$; indeed $OF(F_3)$ is depicted in the following figure.



The simplicial complex $OF(F_3)$

Finally the simplicial complex $OF(F)$ is homotopic to $S^2 \vee S^1$.

4.2. The independence complex. Let $G = (V, E)$ be a graph. The simplicial complex on V whose faces are the subsets of V containing no adjacent vertices is denoted by $\text{Ind}(G)$ and is called the *independence complex* of G . We have

$$(4.1) \quad \begin{aligned} (\text{Ind}(G), v) &= \text{Ind}(G \setminus \{v\}) \\ (\text{Ind}(G) : v) &= \text{Ind}(G \setminus N[v]). \end{aligned}$$

The simplicial complex $\text{Ind}(G)$ is a cone of apex a if and only if a is an isolated vertex of G .

Lemma 4.5. *Let a and b be vertices of G ; a dominates b in $\text{Ind}(G)$ if and only if $N[b] \setminus \{b\} \subset N[a]$.*

Proof. The faces of $\text{Ind}(G \setminus N[a])$ are the independent sets of vertices of $G \setminus N[a]$. Let D be a face of $\text{Ind}(G \setminus N[a])$; $D \cup \{b\}$ is a face of $\text{Ind}(G \setminus a)$ if and only if $b \in D$ or $b \notin N[D]$. Since this must be true for all faces, $N[b] \setminus \{b\} \cap (V \setminus N[a]) = \emptyset$, and the result follows. \square

Lemma 4.6. *Let a be a vertex of G having distance two from a leaf b . Then $\text{Ind}(G)$ collapses onto $\text{Ind}(G \setminus a)$.*

Proof. Since $N[b] \setminus \{b\} \subset N[a]$, a dominates b by Lemma 4.5. Moreover the simplicial complex $\text{Ind}(G \setminus N[a])$ is a cone with apex b , since $G \setminus N[a]$ contains b as an isolated vertex. If $(\sigma_1 \supset \tau_1), \dots, (\sigma_r \supset \tau_r)$ is a sequence of elementary collapses of $\text{Ind}(G \setminus N[a])$ onto \emptyset , then $(\sigma_1 \cup \{a\} \supset \tau_1 \cup \{a\}), \dots, (\sigma_r \cup \{a\} \supset \tau_r \cup \{a\})$ is a sequence of elementary collapses of $\text{Ind}(G)$ onto $\text{Ind}(G \setminus a)$. \square

The removal of vertices at distance two from a leaf has already been used by Kozlov for the independence complex of a path and by Wassmer for rooted forests (see [K1] and [W]).

In a forest F , a vertex a dominates a vertex b if and only if

- (1) b is a leaf and a is adjacent to b ;
- (2) b is a leaf and a has distance two from b ;
- (3) b is isolated.

The third case deals with the trivial case in which $\text{Ind}(F)$ is a cone with apex b . Specifying the treatment of the domination to the first case we obtain the analysis of [MT, Section 6]; specifying it to the second case we obtain the analysis of [W, Section 3.2]. In the first approach what happens is that at each stage the removal of the vertex a and of all its neighbors changes the homotopy type of $\text{Ind}(F)$ by a suspension; thus the relevant informations are the number r_1 of steps required to reach a graph F_1 with no edges and the number i_1 of isolated vertices of F_1 . In the second approach what happens is that at each stage the removal of the vertex a does not change the homotopy type of $\text{Ind}(F)$; thus the relevant informations are the numbers r_2 and i_2 of isolated edges and vertices of the graph F_2 obtained by performing the removal as long as possible. The conclusion is that $i_1 \neq 0$ if and only if $i_2 \neq 0$ if and only if $\text{Ind}(F)$ collapses onto a point. If $i_1 = i_2 = 0$, then $r_1 = r_2 = r$ and $\text{Ind}(F)$ collapses onto the boundary of the r -dimensional cross-polytope; it can be proved that $r = i(F) = \gamma(F)$, see [MT, Theorem 6.4]. We state explicitly the following result for further reference.

Theorem 4.7. *Let F be a forest. Then $\text{Ind}(F)$ is a grape. Moreover, $\text{Ind}(F)$ is either contractible or homotopic to a sphere.*

4.3. The dominance complex. Let $G = (V, E)$ be graph. The simplicial complex on V whose faces are the complements of the dominating sets is denoted by $\text{Dom}(G)$ and is called the *dominance complex of G* ; equivalently the minimal non-faces of $\text{Dom}(G)$ are the minimal elements of $\{N[x] \mid x \in V\}$. The dominance complex of G is never a cone. Let $a \in V$; we have

$$(\text{Dom}(G) : a) = (\text{Dom}(G \setminus a), N[a] \setminus \{a\}).$$

Lemma 4.8. *Let a, b be distinct non-isolated vertices of G ; a dominates b in $\text{Dom}(G)$ if and only if for all $v \in N[b] \setminus N[a]$ there exists $m \in V$ such that $N[m] \setminus \{a\} \subset N[v] \setminus \{b\}$.*

Proof. (\Rightarrow) Let $v \in N[b] \setminus N[a]$ and consider $\sigma := N[v] \setminus \{b\}$. Since $\sigma \cup \{b\} \notin \Delta$ and a dominates b , it follows that $\sigma \notin (\Delta : a)$. Thus there is $m \in V$ such that $N[m] \setminus \{a\} \subset \sigma = N[v] \setminus \{b\}$.

(\Leftarrow) Proceed by contradiction and suppose that a does not dominate b ; hence there exists $\sigma \in (\Delta : a)$ such that $\sigma \cup \{b\} \notin \Delta$. This means that

- (1) $\nexists m \in V$ such that $N[m] \subset \sigma \cup \{a\}$,
- (2) $\exists v \in V$ such that $N[v] \subset \sigma \cup \{b\}$.

If v satisfies $N[v] \subset \sigma \cup \{b\}$, then $N[v] \not\subset \sigma$, since otherwise (1) would not hold. Thus $b \in N[v]$; moreover $a \notin N[v]$, since $N[v] \subset \sigma \cup \{b\}$ and $a \notin \sigma$. Hence $v \in N[b] \setminus N[a]$. By assumption there is $m \in V$ such that $N[m] \setminus \{a\} \subset N[v] \setminus \{b\}$ and hence $N[m] \subset N[v] \cup \{a\} \setminus \{b\} \subset \sigma \cup \{a\}$, contradicting (1). \square

Lemma 4.9. *Let $a, b, c \in V$ and suppose that $N[b] = \{a, b\}$ and $\{a, b, c\} \subset N[a]$. Then $\text{Dom}(G)$ collapses onto $\text{Dom}(G \setminus \text{edge } \{a, c\})$.*

Proof. Thanks to Lemma 4.8 a dominates b since $N[a] \supset N[b]$; $(\text{Dom}(G), a)$ is a cone with apex b . Let $L = (\text{Dom}(G) : N[c] \setminus \{a\}) \subset (\text{Dom}(G), a)$. The simplicial

complex L is a cone with apex b . Let $(\sigma_1 \supset \tau_1), \dots, (\sigma_r \supset \tau_r)$ be a sequence of elementary collapses of L to \emptyset ; adding to σ_i and τ_i the face $N[c] \setminus \{a\}$ for $1 \leq i \leq r$, we obtain a sequence of elementary collapses of $\text{Dom}(G)$ onto the simplicial complex $(\text{Dom}(G), N[c] \setminus \{a\})$. It remains to show that $(\text{Dom}(G), N[c] \setminus \{a\}) = \text{Dom}(G \setminus \text{edge } \{a, c\})$. The minimal non-faces of $(\text{Dom}(G), N[c] \setminus \{a\})$ and $\text{Dom}(G \setminus \text{edge } \{a, c\})$ are respectively the minimal elements of

$$\{N[v] \mid v \in V\} \cup \{N[c] \setminus \{a\}\}$$

and the minimal elements of

$$\{N[v] \mid v \in V \setminus \{a, c\}\} \cup \{N[c] \setminus \{a\}, N[a] \setminus \{c\}\},$$

where by $N[v]$ we mean the closed neighborhood of v in the graph G . Since $N[b] \subset N[a] \setminus \{c\}$, the minimal elements of the two sets above are the same. \square

We now consider the dominance complex of a forest F . Iterating as long as we can the removal of an edge satisfying the conditions of Lemma 4.9, we obtain a subforest F' of F containing only isolated vertices and edges. The forest F' depends on the choices of edges; the number r of edges of F' , though, is independent of the choices thanks to the following result.

Proposition 4.10. *Let F be a forest. Then*

- (1) $\text{Dom}(F)$ is a grape;
- (2) $\text{Dom}(F)$ collapses onto the boundary of an r -dimensional cross-polytope, where r is the number of edges of F' .

Proof. (1) By Lemma 4.8 the vertex a adjacent to a leaf b dominates b , since $N[a] \supset N[b]$. The complex $(\text{Dom}(F), a)$ is a cone with apex b , and $(\text{Dom}(F) : a) = \text{Dom}(F \setminus a)$. Hence the result follows by induction on the number of vertices.

(2) It follows at once from Lemma 4.9 that $\text{Dom}(F)$ collapses onto $\text{Dom}(F')$. Since the dominance complex of F' is the boundary of the cross-polytope of dimension r , where r is the number of edges of F' , the result follows. \square

It can be proved that $r = \alpha_0(F) = \beta_1(F)$ (see [MT, Theorem 8.1]).

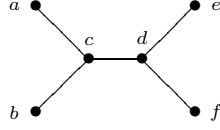
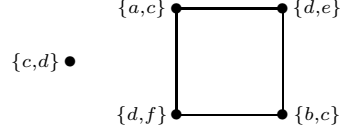
4.4. Matching complex. Let $G = (V, E)$ be a graph. We define a simplicial complex $M(G)$ on E whose faces are the matchings of G , i.e. sets of pairwise disjoint edges. We note that $M(G)$ is the independence complex of the line dual of G , i.e. of the graph whose vertices are the edges of G and where $\{e_1, e_2\}$ is an edge if $e_1 \neq e_2$ and $e_1 \cap e_2 \neq \emptyset$. Note that if $e = \{x, y\} \in E$, then $(M(G), e) = M(G \setminus e)$ and $(M(G) : e) = M(G \setminus \{x\} \setminus \{y\})$.

If F is a forest, then the line dual of F is not a forest unless F is a disjoint union of paths. Hence Theorem 4.7 does not apply to $M(F)$. Nevertheless, we have the following result.

Theorem 4.11. *Let $F = (V, E)$ be a forest. Then $M(F)$ is a grape.*

Proof. We proceed by induction on the number of edges of F , the base case being obvious. Let b be a leaf and let a be adjacent to b . If the edge $\{a, b\}$ is isolated, then $M(F)$ is a cone with apex $\{a, b\}$ and hence it is a grape. Otherwise let $c \neq b$ be adjacent to a . By Lemma 4.5, the edge $\{a, c\}$ dominates the edge $\{a, b\}$ in $M(F)$. The result follows by induction since $(M(F), \{a, c\})$ and $(M(F) : \{a, c\})$ are matching complexes of forests. \square

Example 4.12. The simplicial complex $M(F)$ may be a wedge of spheres of different dimensions. Let F be the tree depicted in the following figure.

The tree F The simplicial complex $M(F)$

The complex $M(F)$ is homeomorphic to $S^1 \vee S^0$.

4.5. Edge covering complex. Let $G = (V, E)$ be a graph. We define a simplicial complex $EC(G)$ on E whose faces are the complements of the edge covers of G . For all $v \in V$, let $\text{star}(v) = \{e \in E \mid v \in e\}$; thus the minimal non-faces of $EC(G)$ are the minimal elements of $\{\text{star}(v) \mid v \in V\}$. Note that if v is an isolated vertex, then $EC(G) = \emptyset$. Let $e = \{x, y\} \in E$; then $(EC(G) : e) = EC(G \setminus e)$ since the minimal non-faces of $(EC(G) : e)$ are the minimal elements of

$$\{\text{star}(v) \mid v \in V, v \neq x, y\} \cup \{\text{star}(x) \setminus \{e\}, \text{star}(y) \setminus \{e\}\}.$$

The complex $EC(G)$ is a cone with apex e if and only if x and y are both adjacent to leaves.

Theorem 4.13. *Let F be a forest. Then $EC(F)$ is a grape. Moreover, $EC(F)$ is either contractible or homotopic to a sphere.*

Proof. We may assume that F has no isolated vertices, since \emptyset is contractible. Proceed by induction on the number of edges of F . If F is a disjoint union of stars, then $EC(F) = \{\emptyset\}$, the (-1) -dimensional sphere. Otherwise, let x_1, \dots, x_4 be distinct vertices such that $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}$ are edges and x_1 is a leaf. If x_4 is a leaf, then $EC(F)$ is a cone with apex $\{x_2, x_3\}$ and we are done. If x_4 is not a leaf, then $\{x_3, x_4\}$ dominates $\{x_2, x_3\}$ since $(EC(F), \{x_3, x_4\})$ is a cone with apex $\{x_2, x_3\}$. Hence $EC(F)$ is homotopic to the suspension of $EC(F \setminus \{x_3, x_4\})$, and we conclude by the inductive hypothesis. \square

The following result relates the simplicial complex $EC(F)$ on E to the simplicial complex $\text{Ind}(F)$ on V . We let $\kappa(F)$ denote the number of connected components of F , or equivalently $\kappa(F) = |V| - |E|$.

Theorem 4.14. *Let F be a forest. Then $EC(F)$ is homotopic to a sphere (resp. contractible) if and only if $\text{Ind}(F)$ is homotopic to a sphere (resp. contractible). Moreover if $EC(F)$ is not contractible, the dimension of the sphere associated to $EC(F)$ is $i(F) - \kappa(F) - 1 = \gamma(F) - \kappa(F) - 1$.*

Proof. We may assume that F has no isolated vertices since in this case $EC(F) = \emptyset$, $\text{Ind}(F)$ is a cone and therefore they are both contractible. Proceed by induction on the number of edges of F . If F is a disjoint union of stars, then $EC(F) = \{\emptyset\}$, the (-1) -dimensional sphere, and $\text{Ind}(F) \simeq S^{\kappa(F)-1}$ (see [MT]). Otherwise, let $x_1, \dots, x_4 \in V$ be such that $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}$ are edges and x_1 is a leaf. If x_4 is a leaf, then $EC(F)$ is a cone with apex $\{x_2, x_3\}$; x_3 dominates x_4 in $\text{Ind}(F)$ and both $(\text{Ind}(F), x_3)$ and $(\text{Ind}(F) : x_3)$ are cones; thus $EC(F)$ and $\text{Ind}(F)$ are both contractible. If x_4 is not a leaf, then $EC(F)$ is homotopic to $\Sigma(EC(F'))$, where $F' = F \setminus \text{edge } \{x_3, x_4\}$, while $\text{Ind}(F)$ is homotopic to $\text{Ind}(F')$ by Lemma 4.6.

By the inductive hypothesis we have that $EC(F')$ and $\text{Ind}(F')$ are either both contractible or both homotopic to spheres and thus also $EC(F)$ and $\text{Ind}(F)$ have the same property. Moreover if $EC(F)$ is not contractible, then it is homotopic to a sphere of dimension $\gamma(F') - \kappa(F') = \gamma(F) - \kappa(F) - 1$. The equalities $i(F) = i(F')$ and $\gamma(F) = i(F)$, when $EC(F)$ and $\text{Ind}(F)$ are not contractible, follow from [MT, Theorem 6.4]. \square

4.6. Edge dominance complex. Let $G = (V, E)$ be a graph. We define a simplicial complex $ED(G)$ on E whose faces are the complements of the dominating sets of the line dual of G . For all $e \in E$, let $\text{star}(e) = \{f \in E \mid f \cap e \neq \emptyset\}$; thus the minimal non-faces of $ED(G)$ are the minimal elements of $\{\text{star}(e) \mid e \in E\}$.

Theorem 4.15. *Let F be a forest. Then $ED(F)$ is a grape. Moreover $ED(F)$ is homotopic to a sphere of dimension $|E| - \beta_1(F) - 1 = |E| - \alpha_0(F) - 1$.*

Proof. Proceed by induction on the number of edges of F . If F consists only of isolated vertices and edges, then $ED(F) = \{\emptyset\}$, the (-1) -dimensional sphere, and the result is clear. Let b be a leaf of F and let $\{a, c\}$ be an edge of F such that a is adjacent to b and $c \neq b$. Since $\text{star}(\{a, b\}) \subset \text{star}(\{a, c\})$, we deduce from Lemma 4.8 that $\{a, c\}$ dominates $\{a, b\}$. The complex $(ED(F), \{a, c\})$ is a cone with apex $\{a, b\}$. Since $(ED(F) : \{a, c\}) = ED(F \setminus \text{edge } \{a, c\})$ and $ED(F) \simeq \Sigma(ED(F) : \{a, c\})$, we conclude by induction that $ED(F)$ is a grape and that it is homotopic to a sphere. To compute the dimension of the sphere, let $M \subset E$ be a matching of maximum cardinality and b be a leaf adjacent to the vertex a . We may assume that the edge $\{a, b\}$ is not isolated. If $\{a, b\} \in M$, then removing an edge $\{a, c\}$ with $c \neq b$ we may conclude by induction. If $\{a, b\} \notin M$, then an edge $\{a, c\} \in M$ for exactly one c . The set $M \cup \{a, b\} \setminus \{a, c\}$ is again a matching with same cardinality as M , and we may conclude as before. The last equality follows by a similar argument or by Theorem 2.1. \square

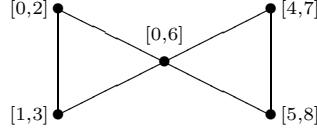
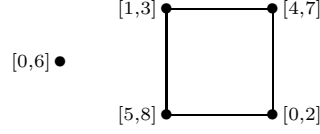
4.7. Interval order complex. Let X be a finite set of closed bounded intervals in \mathbb{R} ; the interval order complex on X is the simplicial complex $\mathcal{O}(X)$ whose faces are the subsets of X consisting of pairwise disjoint intervals. The simplicial complex $\mathcal{O}(X)$ is shellable thanks to [BM]. In particular, it follows that $\mathcal{O}(X)$ is contractible or homotopic to a wedge of spheres. We give a short direct computation of the homotopy type of $\mathcal{O}(X)$.

Associated to X there is also a graph $O(X) = (V, E)$, where $V = X$ and $\{I, J\} \in E$ if and only if $I \cap J \neq \emptyset$. Clearly, $\text{Ind}(O(X)) = \mathcal{O}(X)$. Theorem 4.7 does not apply to $\text{Ind}(O(X))$, since $O(X)$ is not in general a forest. Nevertheless we have the following result.

Theorem 4.16. *The simplicial complex $\mathcal{O}(X)$ is a grape.*

Proof. If $X = \emptyset$, then the result is clear. Otherwise let $I = [a, b] \in X$ be an interval such that $b = \min\{y \mid [x, y] \in X\}$. The vertices of $O(X)$ adjacent to I are the intervals of X containing b . If no interval in $X \setminus \{I\}$ contains b , then $\mathcal{O}(X)$ is a cone with apex I and we are done. Otherwise, let $J \in X$ be an interval containing b . By construction we have $N[I] \subset N[J]$ (in the graph $O(X)$) and by Lemma 4.5 we deduce that J dominates I in $\mathcal{O}(X)$. Since $(\mathcal{O}(X), J) = \mathcal{O}(X \setminus \{J\})$ and $(\mathcal{O}(X) : J) = \mathcal{O}(X \setminus N[J])$, we conclude by induction on the cardinality of X . \square

Example 4.17. The simplicial complex $\mathcal{O}(X)$ may be a wedge of spheres of different dimensions. Let $X = \{[0, 2], [0, 6], [1, 3], [4, 7], [5, 8]\}$. The graph $O(X)$ and the simplicial complex $\mathcal{O}(X)$ are depicted in the following figure.

The graph $O(X)$ The simplicial complex $\mathcal{O}(X)$

The complex $\mathcal{O}(X)$ is homeomorphic to $S^1 \vee S^0$.

We summarize in the following table the results obtained in this section on the homotopy types of the simplicial complexes associated to a (possibly multidirected) forest $F = (V, E)$ and of the interval order complex. Wedge of spheres means that the spheres have in general different dimensions and the wedge could be empty (i.e. the simplicial complex could be contractible).

Name	Homotopy type
Oriented forests	Wedge of spheres
Independence complex	Contractible or sphere of dimension $i(F) - 1 = \gamma(F) - 1$
Dominance complex	Sphere of dimension $\alpha_0(F) - 1 = \beta_1(F) - 1$
Matching complex	Wedge of spheres
Edge covering complex	Contractible or sphere of dimension $ E - V + i(F) - 1 = E - V + \gamma(F) - 1$
Edge dominance complex	Sphere of dimension $ E - \alpha_0(F) - 1 = E - \beta_1(F) - 1$
Interval order complex	Wedge of spheres

REFERENCES

- [AL] Allan, R.B., Laskar, R., *On domination and independent domination numbers of a graph*, Discrete Mathematics, 23 (1978), 73-76.
- [ALH] Allan, R.B., Laskar, R., Hedetniemi S., *A note on total domination*, Discrete Mathematics, 49 (1984), 7-13.
- [BK] Babson, E., Kozlov, D., *Proof of the Lovász Conjecture*, Ann. of Math., to appear.
- [BM] Billera, L., Myers, A., *Shellability of interval orders*, Order 15 (1998/99), no. 2, 113-117.
- [B] Bollobás, B., *Modern Graph Theory*, Graduate Texts in Mathematics, Vol. 184, Springer, 1998.
- [BC] Bollobás, B., Cockayne, E.J., *Graph-theoretic parameters concerning domination, independence, and irredundance*, J. Graph Theory 3 (1979), no. 3, 241-249.
- [BLN] Bousquet-Mélou, M., Linusson, S., Nevo, E., *On the independence complex of square grids*, arXiv:math/0701890v2 [math.CO].
- [C] Cohen, M.M., *A course in simple-homotopy theory*, Graduate Texts in Mathematics, Vol. 10, Springer-Verlag, New York-Berlin, 1973.
- [D] Diestel, R., *Graph Theory*, Graduate Texts in Mathematics, Vol. 173, Springer, 1997.
- [EH] Ehrenborg, R., Hetyei, G., *The topology of the independence complex*, European J. Combin. 27 (2006), no. 6, 906-923.

- [E] Engström, A., *Complexes of Directed Trees and Independence Complexes*, arXiv:math/0508148v1 [math.CO].
- [ET] Erdős, P., Tuza, Z., *Vertex coverings of the edge set in a connected graph*, Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992), 1179-1187, Wiley-Intersci. Publ., Wiley, New York, 1995.
- [HHS] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208. Marcel Dekker, Inc., New York, 1998.
- [HY] Henning, M.A., Yeo, A., *Total domination and matching numbers in claw-free graphs*, Electron. J. Combin. 13 (2006), no. 1, Research Paper 59, 28 pp.
- [Ha] Hatcher, A., *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [J] Jonsson, J., *Hard squares with negative activity and rhombus tilings of the plane*, Electron. J. Combin. 13 (2006), no. 1, Research Paper 67, (electronic).
- [Kl] Klivans, C., *Threshold graphs, shifted complexes, and graphical complexes*, arXiv:math/0703114v1 [math.CO].
- [K1] Kozlov, D., *Complexes of directed trees*, J. Combin. Theory Ser. A 88 (1999), no. 1, 112-122.
- [K2] Kozlov, D., *Directed trees in a string, real polynomials with triple roots, and chain mails*, Discrete Comput. Geom. 32 (2004), no. 3, 373-382.
- [MT] Marietti, M., Testa, D., *Cores of simplicial complexes*, arXiv:math/0703351v1 [math.CO].
- [W] Wassmer, A., *A Dual Independence Complex*, PhD Thesis, Technische Universität Berlin, 2005.

UNIVERSITÀ DEGLI STUDI DI ROMA “LA SAPIENZA”, PIAZZALE A. MORO 5, 00185 ROMA, ITALY
E-mail address: `marietti@mat.uniroma1.it` `www.mat.uniroma1.it/~marietti`

UNIVERSITÀ DEGLI STUDI DI ROMA “LA SAPIENZA”, PIAZZALE A. MORO 5, 00185 ROMA, ITALY
E-mail address: `testa@mat.uniroma1.it` `www.mat.uniroma1.it/~testa`